Hamilton-Jacobi-Bellman-Isaacs equations in Hilbert spaces with applications in decision making under uncertainty

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A typical (open) neighbourghood ...

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Fundamental questions:

How do we make decisions in stochastic models under uncertainty concerning the exact stochastic model governing the system?

If the state of the system is distributed in “space” (e.g. as in systems related to spatial economics, resource management or other problems related to the physical world) how can be formulate spatial decision rules which are robust under model uncertainty?

How can the theory of nonlinear PDEs in infinite dimensional spaces interact with stochastic analysis and provide us with a concrete framework for the treatment of such problems?

We will present a general framework which allows us to express spatially dependent decision problems under model uncertainty as an infinite dimensional stochastic differential game.

Using dynamic programming techniques we will show that the value function of the game satisfies a nonlinear PDE on an infinite dimensional Hilbert space, the solution of which if it exists, will provide the value function under various scenarios concerning the initial state of the system.

We will then show existence of a weak type of solutions for this equation (mild solutions) and connect them with the construction of robust optimal controls for the system.

These solutions allow us to obtain important information concerning spatial variability and uncertainty.

Motivation : Spatial Resource Economics

Consider spatially distributed biomass in an open connected bounded subset $\mathcal{O} \subset \mathbb{R}^d$ with sufficiently smooth boundary, representing geographical space, e.g. a fishery, and let $y(t,z)$ be the biomass density at time t and point $z \in \mathcal{O}$.

The spatiotemporal evolution of the biomass density is described by the reaction diffusion type population dynamics equation

$$\frac{\partial}{\partial t} y(t,z) = D\Delta y(t,z) + f(y(t,z)) - u(t,z) + \dot{W}(t,z), \qquad (1)$$

where by Δ we denote the Laplace operator (with respect to the variables z), $f : \mathbb{R} \to \mathbb{R}$ corresponds to a reaction term modelling the local population dynamics, $u(t,z) > 0$ is the density of harvesting activity at time t and point $z \in \mathbb{R}^d$, while $\dot{W}(t,z)$ corresponds to spatiotemporal stochastic fluctuations.

The above equation will be supplemented with an initial condition $y(0, z) = y_0(z)$ for some suitable function $y_0 \in L^2(\mathcal{O})$, and a suitable boundary condition, which for the sake of this example is assumed to be either homogeneous Dirichlet or periodic.

The Laplace operator is chosen to model the transport of biomass in space; clearly another similar operator, perhaps including an advection field.

Concerning the function f, modelling population dynamics, a number of choices are possible, one may consider an affine form $f(s) = a_0 + a_1 s$, a logistic form $f(s) = a_0 s(1 - \frac{s}{N})^{a_1}$, for suitable constants a_0, a_1 where N corresponds to the carrying capacity of the population, or even nonlocal forms such as for instance $f[y](z) = f((Sy)(z))$ where S is an integral operator of the form $[Sy](z) = \int_{\mathcal{O}} k(z, z')y(z')dz'$ for a suitable kernel k, modeling some sort of local averaging effect.

If the probability law for the fluctuations is known the fishery manager will choose the harvesting protocol u such that she maximizes intertemporal benefits from the harvest, while keeping the resource at a required level for sustainability of the fishery, hence tries to choose u so as to maximize

$$J(u) = \mathbb{E}_P[\int_0^\infty e^{-\delta t} \int_{\mathcal{O}} \left\{ g(y(t,z)) + \frac{1}{1-\nu}(u(t,z))^{1-\nu} \right\} dzdt],$$

under the state constraint (24).

g is a benefit function which models the benefits from the remaining biomass after harvesting.

In this case a dynamic programming approach can be used to provide a framework for the derivation of the optimal harvesting protocol.

What if the stochastic model for the fluctuations is not known exactly and more than one stochastic models may be used to decribe them?

For instance what if $\dot{W}(t,z)$ is not exactly a Wiener process but allows for spatio-temporal dependent drift $\upsilon(t,z)$?

If we design our control procedure u under model P (i.e. equiv. under the assumption the $\upsilon(t,z) = 0$) and this is not the case, then clearly our control procedure will underperform leading to possibly catastrophic results.

Can we design an optimal control procedure which will be robust under such model uncertainty?

Robust control theory

Suppose that given that a model P can describe well the behaviour of a system and given that a control procedure u is adopted the performance of the system is given by $J(u; P) = \mathbb{E}_P[J(u)]$.

If we are sure about the probability model P then we simply choose u so as to solve $\max_u \mathbb{E}_P[J(u)]$.

Suppose now that there is a family of alternative models $\mathcal{Q}$ which may describe the phenomenon.

One way to make a decision is to use a minimax criterion (Gilboa-Schmeidler) and solve the problem

$$\max_u \min_{P \in \mathcal{Q}} \mathbb{E}_P[J(u)]$$

This will give you a decision that will keep you even at the worst case scenario for the choice of P.

However, it may be that some of the probability models in $\mathcal{Q}$ are not as highly possible as some others.

For instance some of these may correspond to very extreme scenarios.

To this end we may choose to penalize some of these probability laws, using a convex penalty term $a : \mathcal{M}(\Omega) \to \mathbb{R}$ which takes very large values for the improbable models.

We may then try to make a decision using

$$\max_{u} \min_{P \in \mathcal{M}(\Omega)} [\mathbb{E}_P[J(u)] + a(P)].$$

This approach has become very popular in economic theory recently.

Hansen and Sargent (Nobel Prize Economics 2013, 2011) developed a theory for economic problems in the temporal domain along these lines using the Kulback-Leibler entropy for a.

Marinacci and coworkers, have developed a general axiomatic framework for decision making for general convex function a, called variational preferences.

In the context of risk management similar ideas have been formulated by Arzner and Fölmer and Shied, leading to the development of the concept of convex risk measures which is gaining a lot of popularity in both academia and business.

These ideas have been extended in spatially extended systems by Brock, Xepapadeas and Yannacopoulos, requiring techniques from the theory of infinite dimensional stochastic dynamical systems.

Importantly the minimax nature of the decision theoretic model gives it a game theoretic flavour, where player A (decision maker) plays against a malevolent player B (nature) and the robust decision rule can be expressed as a Nash equilibrium.

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The robust control model

Consider the following controlled system in $\mathbb{H}$:

$$\begin{aligned} dX(t) &= (AX(t) + F(X(t)) + Bu(t))dt + CdW(t) \\ X(0) &= x \in \mathbb{H}, \end{aligned} \tag{2}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a possibly unbounded linear operator, $B : \mathbb{H} \to \mathbb{H}$ and $C : \mathbb{H} \to \mathbb{H}$ are linear operators, $F : \mathbb{H} \to \mathbb{H}$ is a non-linear mapping and W is a $\mathbb{H}$-valued cylindrical Wiener process, with respect to the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$

The control procedure $u(\cdot)$ is assumed to be an $\mathbb{H}$-valued process such that $u(\cdot) \in \mathcal{U}_{ad}$ where $\mathcal{U}_{ad}$ is the set of all admissible control processes. A possible choice for $\mathcal{U}_{ad}$ is the set of all processes $u(\cdot) \in \mathcal{H} := L^2_{\mathcal{F}_t}(0, +\infty; \mathbb{H})$ such that $|u(s)|_{\mathbb{H}} \leq K_\alpha$ for some $K_\alpha > 0$ and every $s \geq 0$, but of course other choices are also possible.

Definition

We say that the Cauchy problem (2) admits a mild solution, if and only if there exists a $\mathcal{F}_t$ measurable and square integrable stochastic process X with values in $\mathbb{H}$, such that

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}\Big[F(X(s)) + Bu(s)\Big]ds + \int_0^t e^{(t-s)A}CdW(s).$$

We also need to define the generator operator for the uncontrolled $(B = 0)$ state process (2)

Definition

The operator $L_F : D(L) \subset C_b(\mathbb{H}) \to C_b(\mathbb{H})$ defined by

$$L_F V(x) := \langle Ax + F(x), DV(x)\rangle_{\mathbb{H}} + \frac{1}{2}\mathrm{Tr}\left[CC^* D^2 V(x)\right], \qquad (3)$$

will be called the generator operator for the uncontrolled state process (2), $(B = 0)$.

Let us now envision a decision maker who controls the system (2) by choosing the stochastic process $u(\cdot) \in \mathcal{U}_{ad}$.

If the agent was certain that the stochastic process $X(t)$ is indeed modelled by the probability law induced on $\mathbb{H}$ by the state equation (2) (or equivalently that the reference probability model $\mathbb{P}$ can indeed be trusted), she could try to choose the control process $u \in \mathcal{U}_{ad}$ optimally so as to solve a maximization problem of the form

$$\sup_{u(\cdot)\in\mathcal{H}} \mathbb{E}_{\mathbb{P}} \left[\int_0^\infty e^{-\delta t} \left(g(X(t)) + K(u(t)) \right) dt \right], \tag{4}$$

where $\delta > 0$, is a discount factor, $g : \mathbb{H} \to \mathbb{R}$ is a utility function modelling the instantaneous satisfaction derived by the agent if the system is in state $X(t)$ at time t and $K : \mathbb{H} \to \mathbb{R}_+$ is a profit function associated with the choice of the control process.

g, K strictly concave and usc functions, $g \in UC_b(\mathbb{H})$, including a suitable penalization term which ensures that $u(\cdot) \in \mathcal{U}_{ad} \subset \mathcal{H}$.

We assume that the decision maker is uncertain about the true model $\mathbb{P}$ under which the state process follows the law induced by (2).

In this framework, there is a set $\mathcal{P}$ of possible probability models for the law of the state $X(t)$, which if the data of the problem $(A, F, B, C, X(0))$, are assumed to be well specified can be attributed to possible mispecifications of the stochastic factor term W introducing stochastic fluctuations into the state equation (2).

A plausible strategy for the decision maker would be to solve a robust optimal contol problem of the form

$$\sup_{u(\cdot)\in\mathcal{H}} \inf_{\widetilde{\mathbb{P}}\in\mathcal{P}} \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\int_0^\infty e^{-\delta t}\left(g(X(t)) + K(u(t))\right) dt\right], \tag{5}$$

where now the agent chooses to maximize the output for the worst case scenario, concerning the stochastic factor, out of all plausible probability models $\widetilde{\mathbb{P}} \in \mathcal{P}$.

Clearly, the dynamic constraint (2) will have to be modified accordingly, taking into account the particular probability model $\widetilde{\mathbb{P}}$ chosen by the agent.

We will restrict our attention to models which are absolutely continuous with respect to a reference probability measure $\mathbb{P}$. We further parameterize the density in terms of a Hilbert space valued stochastic process $\upsilon(\cdot)$, taking values in a subset $\mathcal{V}_{ad} \subset \mathcal{H}$ in terms of the exponential process

$$\mathcal{M}(t) = \exp\left(\int_0^t \langle \upsilon(s), dW(s) \rangle_{\mathbb{H}} - \frac{1}{2} \int_0^t |\upsilon(s)|^2_{\mathbb{H}} ds \right), \quad t > 0, \quad (6)$$

with $\mathcal{V}_{ad}$ chosen so as to guarantee the integrability of the process and its martingale properties (usually expressed in terms of the Novikov condition).

The set of allowable probability measures is then characterized by the allowed $\mathbb{H}$-valued processes $\upsilon(\cdot)$, in the sense that there is an one to one correspondence between any stochastic process $\upsilon(\cdot) \in \mathcal{V}_{ad}$ and any probability measure $\widetilde{\mathbb{P}} \in \mathcal{P}$, denoted by $\widetilde{\mathbb{P}}(\upsilon(\cdot))$, with $\upsilon(\cdot) = 0$ corresponding to the reference measure.

If $\upsilon(\cdot)$ deviates from zero, by the representation (6) for the exponential density we note that the chosen measure $\widetilde{\mathbb{P}}$ will deviate from the reference measure, so that large values of $\upsilon(\cdot)$ will extend the set $\mathcal{P}$ of allowable models.

This is bound to cause some discomfort to an uncertainty averse agent, who will try to penalize such values of υ by a penalty function $\mathcal{T} : \mathcal{H} \to \mathbb{R}$ which will be chosen to be strictly convex and lower semicontinuous, and such that $\upsilon(\cdot) \in \mathcal{V}_{ad} \subset \mathcal{H}$.

We assume a general penalty function of the form

$$T(\upsilon(\cdot)) = \int_0^\infty e^{-\delta t} T_\ell(\upsilon(s))ds, \tag{7}$$

where $T_\ell : \mathbb{H} \to \mathbb{R}$ is a strictly convex lower semicontinuous function. The special choice $T(\upsilon) := \frac{1}{2}|\upsilon|^2_{\mathbb{H}}$ corresponds to the case of entropic constraints.

Assuming that a choice of $\upsilon(\cdot)$ has been made by the agent, then a corresponding probability measure $\widetilde{\mathbb{P}}$, denoted by $\widetilde{\mathbb{P}}(\upsilon(\cdot))$ to emphasize the dependence on the choice of $\upsilon(\cdot)$, has been selected as a possible model for the state process (2).

Using the Girsanov theorem the controlled system (2) under the equivalent probability measure $\widetilde{\mathbb{P}} \in \mathcal{P}(\upsilon(\cdot))$ satisfies the SDE in $\mathbb{H}$:

$$\begin{aligned} dX(t) &= (AX(t) + F(X(t)) + Bu(t) + C\upsilon(t))dt + Cd\widetilde{W}(t) \\ X(0) &= x \in \mathbb{H}. \end{aligned} \tag{8}$$

where is $\left\{\widetilde{W}(t),\ t \geq 0\right\}$ is a $\mathbb{H}$-valued $(\mathcal{F}, \widetilde{\mathbb{P}})$ cylindrical Wiener process, the mild solution of which will be denoted by $X(t; u, \upsilon)$.

This implies that given the agent trusts the probability model $\widetilde{\mathbb{P}} \in \mathcal{P}(\upsilon(\cdot))$, will solve the optimal control problem

$$\sup_{u(\cdot)\in\mathcal{H}} \mathbb{E}_{\widetilde{\mathbb{P}}(\upsilon)} \left[\int_0^\infty e^{-\delta t} \left(g(X(t)) + K(u(t)) \right) dt \right], \qquad (9)$$

subject to dynamic constraints given by (8).

Being uncertain about the choice of $\upsilon(\cdot)$, she must then scan all available choices for $\upsilon(\cdot)$, as allowed by the penalty function $\mathcal{T}$, or equivalently T, and choose a policy u that will maximize this utility function assuming the worst possible scenario over all possible $\upsilon(\cdot)$.

This leads to the minimax problem

$$\sup_{u(\cdot)\in\mathcal{H}} \inf_{\upsilon(\cdot)\in\mathcal{H}} J(x; u(\cdot), \upsilon(\cdot)) :=$$

$$\sup_{u(\cdot)\in\mathcal{H}} \inf_{\upsilon(\cdot)\in\mathcal{H}} \mathbb{E}_{\widetilde{\mathbb{P}}(\upsilon(\cdot))} \left[\int_0^\infty e^{-\delta t} \left(g(X(t)) + K(u(t)) + \theta T(\upsilon(t)) \right) dt \right], \tag{10}$$

subject to the dynamic constraint (8), where θ denotes the preference for the robustness parameter, and the function T may also include a penalty term so as to ensure that $\upsilon(\cdot) \in \mathcal{V}_{ad} \subset \mathcal{H}$, in the standard practice used in convex analysis.

This problem is in the form of a zero sum stochastic differential game with two players: the decision maker (choosing u) and a fictitious adversarial agent, who is commonly referred to, as Nature (choosing υ).

The value of the game (in the Nash sense), if it exists, is given by the value function $\mathcal{V} : \mathbb{H} \to \mathbb{R}$ defined by

$$\mathcal{V}(x) := \sup_{u(\cdot)\in\mathcal{H}} \inf_{\upsilon(\cdot)\in\mathcal{H}} J(x; u, \upsilon). \tag{11}$$

We say that a control pair $(u^*(\cdot), \upsilon^*(\cdot)) \in \mathcal{H} \times \mathcal{H}$ is an optimal control pair, if

$$\mathcal{V}(x) = J(x; u^*(\cdot), \upsilon^*(\cdot)),$$

and that it is a saddle point equilibrium, if

$$J(x; u(\cdot), \upsilon^*(\cdot)) \leq J(x; u^*(\cdot), \upsilon^*(\cdot)) \leq J(x; u^*(\cdot), \upsilon(\cdot)), \quad \forall\, (u(\cdot), v(\cdot)) \in \mathcal{H} \times \mathcal{H}.$$

It is well known that for some cases the value function for a stochastic differential game can be characterized in terms of a non-linear partial differential equations known as the Hamilton-Jacobi-Belman-Isaacs (HJBI) equation (see e.g. Fleming and Souganidis).

The study of the infinite dimensional stochastic differential game (10) (related to the robust control problem (5)) in terms of the corresponding infinite dimensional HJBI equation is the main objective of this paper.

In particular, in the rest of the paper, our plan is to (i) derive the HJBI equation associated with the stochastic differential game (10) (ii) prove that the HJBI equation admits a unique solution (in some appropriate sense) (iii) prove that this solution coincides with the value function of the problem (11) and finally (iv) to derive, in feedback form, an optimal control pair for the problem which is also a saddle point equilibrium.

Standing Assumptions

Assumption

We shall assume that:

(i). $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a densely defined linear operator, generator of a strongly continuous semigroup of contractions $\{e^{tA},\ t \geq 0\}$ on $\mathbb{H}$.

(ii) $B, C \in \mathcal{L}(\mathbb{H})$, $e^{sA}CC^* e^{sA^*} \in \mathcal{L}_1(\mathbb{H})$, for every $s > 0$, while $\int_0^t Tr(e^{sA}CC^* e^{sA^*})ds$ is of trace class for every $t \geq 0$ so that the family of operators $\{Q_t \ : \ t \geq 0\}$ defined by

$$Q_t := \int_0^t e^{sA}CC^* e^{sA^*} ds, \tag{12}$$

is of trace class for every $t \geq 0$.

(iii) $e^{tA}(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$ for every $t > 0$ so thatthe operator $\Gamma(t) := Q_t^{-1/2} e^{tA} \in \mathcal{L}(\mathbb{H})$ and we further assume that the function $t \mapsto \|\Gamma(t)\|$ is integrable in a right neighbourghood of 0 (e.g. $\|\Gamma(t)\| \leq c \max(1, t^{-\rho})$, $t > 0$, $\rho > 0$).

Assumption

(iv). $F : \mathbb{H} \to \mathbb{H}$ *is non-linear mapping such that* $F \in C_b^1(\mathbb{H};\mathbb{H})$, *whereas* $g : \mathbb{H} \to \mathbb{R}$ *is a strictly concave function, with* $g \in UC_b(\mathbb{H})$.

(v). $K : \mathbb{H} \to \mathbb{R}$ *is a strictly concave function, with concave conjugate* $K_\star : \mathbb{H} \to \mathbb{R}$ *defined by*

$$K_\star(x^*) := \inf_{x\in\mathbb{H}} \Big[\langle x^*, x\rangle_{\mathbb{H}} - K(x) \Big], \text{ for every } x^* \in \mathbb{H}, \tag{13}$$

being a Lipschitz function with Lipschitz constant $\|K_\star\|_L$.

(vi). $T : \mathbb{H} \to \mathbb{R}$ *is a strictly convex function, and for any* $\theta > 0$ *we define the family of modified convex conjugate* $T_\theta^* : \mathbb{H} \to \mathbb{R}$ *defined by*

$$T_\theta^\star(x^*) := \sup_{x\in\mathbb{H}} \Big[\langle x^*, x\rangle_{\mathbb{H}} - \theta T(x) \Big], \text{ for every } x^* \in \mathbb{H}, \tag{14}$$

being a Lipschitz function, with Lipschitz constant $\|T_\theta^\star\|_L$. *For* $\theta = 1$ *the modified convex conjugate function coincides with the standard definition of the convex conjugate,* T^*.

The HJBI equation

We are now in position to derive the HJBI equation associated with the stochastic differential game (10), which in our case is an elliptic partial differential equation on the infinite dimensional separable Hilbert space $\mathbb{H}$, by applying standard dynamic programming techniques.

Theorem

The HJBI equation associated with the stochastic differential game (10) is the infinite dimensional elliptic partial differential equation

$$\delta V(x) = L_F V(x) - K_\star\left(-B^* DV(x)\right) - T_\theta^\star\left(-C^* DV(x)\right) + g(x), \quad x \in \mathbb{H}, \tag{15}$$

where L_F is the linear operator defined in (3) *and $K_\star$ and $T_\theta^\star$ are as in* (13) *and* (14) *respectively.*

(Skech of proof) Assuming for the time being sufficient regularity of the value function, it is well known by the theory of stochastic differential games that the value function can be characterized in terms of the HJBI equation

$$\delta V(x) = \sup_{u \in \mathbb{H}} \inf_{\upsilon \in \mathbb{H}} H(x, V(x); u, \upsilon), \tag{16}$$

where H is the pre-Hamiltonian defined by

$$H(x, V(x); u, \upsilon) = L_F V(x) + H_1(V(x); u) + H_2(V(x); \upsilon) + g(x),$$

where

$$\begin{aligned} H_1(V(x); u) &:= \langle Bu, DV(x) \rangle_{\mathbb{H}} + K(u) \\ H_2(V(x); \upsilon) &:= \langle C\upsilon, DV(x) \rangle_{\mathbb{H}} + \theta T(\upsilon). \end{aligned}$$

We then perform the static optimizations required.

The smoothness condition referred to in the proof of the theorem are needed for the application of Itō's lemma, which allows us to pass from the dynamic programming principle to the HJBI equation (16), which are typically second order continuous differentiability conditions.

Under these a priori assumptions on the value function V, the theorem guarantees that the value function will be a solution of the infinite dimensional elliptic equation (15).

However, typically elliptic equations of the form (15) do not admit smooth solutions, for general data, thus posing some questions regarding Theorem 5.

For this reason a variety of weaker solutions for equations of the form (15) can be defined, such as for instance viscosity (typically just continuous solutions) or mild solutions (typically just once continuously differentiable solutions), and it may be shown that eventhough the derivation in the proof of the Theorem is formal, still the weak solution of the HJBI is the value function of the stochastic differential game.

Existence of mild solutions for the HJBI equation

Our main result is the following:

Theorem

Let the standing assumptions hold. Define the family of functions $\mathbb{F}_\theta : \mathbb{H} \to \mathbb{R}$ *by* $\mathbb{F}_\theta(p) := K_\star(B^* p) + T^\star_\theta(C^* p)$, *and let*

$$\theta_0 := \inf\{\theta > 0 \ : \ \partial^+ \mathbb{F}_\theta(p) \neq \emptyset, \ \forall\, p \in \mathbb{H}, \ \text{and such that} \ \partial^+ \mathbb{F}_\theta(p) \ \text{bounded}\},$$

where $\partial^+ \mathbb{F}_\theta(p)$ *denotes the superdifferential of the function* $\mathbb{F}_\theta$ *at* $p \in \mathbb{H}$, *assumed to be uniformly bounded in* θ.
Then, for $\theta > \theta_0$ *the HJBI equation* (15) *admits a unique mild solution in* $UC^1(\mathbb{H})$ *for all* $\delta > 0$.
If $\theta < \theta_0$ *then the HJBI equation* (15) *admits a unique mild solution in* $UC^1(\mathbb{H})$ *for all* $\delta > \delta_0$, *where the critical value*

$$\delta_0 := \left(c\,\|F\|_0 + \|B^*\|\,\|K_\star\|_L + \|C\| \sup_{\theta > \theta_0} \|T^\star_\theta\|_L \right)^{\frac{2}{1-\gamma}}.$$

Sketch of proof

Step 1: Show existence of solution for large δ using a contraction mapping principle in $UC_b^1(\mathbb{H})$.

Use the linear elliptic equation

$$(\delta I - L)u = \psi, \;\; \psi \in UC_b(\mathbb{H}),$$

as starting point and rewrite the HJBI as a nonlinear integral equation

$$\phi = R(\delta, L)(\langle F, D\phi \rangle_{\mathbb{H}} + \sum_{i=1}^{2} F_i(D\phi) - g), \qquad (17)$$

where $R(\delta, L)\psi = u$ is the solution of $(\delta I - L)u = \psi$.

$R(\delta, L)$ is the resolvent operator of the so called Ornstein-Uhlenbeck (O-U) transition semigroup, defined as

$$[P_t\varphi](x) := \mathbb{E}_{\mathbb{P}}\left[\varphi(e^{tA}x + \int_0^t e^{(t-s)A}CdW(t))\right]$$

$$= \int_{\mathbb{H}} \varphi\Big(e^{tA}x + y\Big)\mathcal{N}_{Q_t}(dy), \quad \text{for any } \varphi \in UC_b(\mathbb{H}),$$

where $\{W(t),\ t \geq 0\}$ is a Wiener process under the measure $\mathbb{P}$, and $\mathcal{N}_{Q_t}$ is a centered Gaussian measure on $\mathbb{H}$ with covariance operator Q_t .

This semigroup enjoys some very important properties, fundamental for the study of the HJBI

Proposition

Under the standing assumptions it holds that:

(i) $\psi := R(\delta, L)\varphi = \int_0^\infty e^{-\delta t} P_t\varphi(x)dt$ *is the unique solution to the infinite dimensional elliptic system* $(\delta I - L)\psi = \varphi$.

(ii) *For any* $\varphi \in B_b(\mathbb{H})$ *and any* $t > 0$, *we have that* $P_t\varphi \in UC_b^\infty(\mathbb{H})$, *and in particular for any* $n \in \mathbb{N}$ *there exists a constant* $c_n > 0$ *such that* $\|D^n P_t\varphi(x)\| \leq c_n \|Q_t^{-1/2} e^{tA}\|^n \|\varphi\|_0$, $t > 0$, $x \in \mathbb{H}$.

(iii) *For any* $\delta > 0$, *there exists some positive constant* c *such that*

$$|DR(\delta, L)\varphi(x)|_{\mathbb{H}} \leq cf(\delta)\,\|\varphi\|_0\,, \tag{18}$$

where $f(\delta) = \delta^{\frac{\gamma-1}{2}}$ *and* $\gamma \in [0, 1)$.

Use the new variable $\psi := \delta\phi - L\phi$, express the HJBI in terms of

$$\psi - \langle F, DR(\delta, L)\psi\rangle_{\mathbb{H}} - \sum_{i=1}^{2} F_i(DR(\delta, L)\psi) + g = 0,$$

which may be written as the nonlinear operator equation

$$\psi - \Gamma_\delta \psi = 0$$

and construct the fixed point scheme

$$\psi^{(m+1)} - \langle F, DR(\delta, L)\psi^{(m)}\rangle_{\mathbb{H}} - \sum_{i=1}^{2} F_i(DR(\delta, L)\psi^{(m)}) + g = 0$$

By the smoothing properties of the OU-semigroup, this maps $UC_b^1(\mathbb{H})$ into itself and for $\delta > \delta_0$ this is a contraction, hence a unique fixed point – which is a mild solution of HJBI – exists.

Step 2: Continue this solution for any $\delta > 0$.

This requires techniques from nonlinear analysis, and in particular the theory of maximal monotone operators.

Let $\mathbb{X}$ be a Banach space, $\mathbb{X}^*$ its dual space, by $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathbb{X}$ and $\mathbb{X}^*$ and by $J : \mathbb{X} \to 2^{\mathbb{X}^*}$ denote the duality mapping, i.e. the possibly multivalued nonlinear mapping defined by $J(x) = \{x^* \in \mathbb{X}^* \mid \langle x^*, x \rangle = \|x\|^2_{\mathbb{X}} = \|x^*\|^2_{\mathbb{X}^*}\}$.

Definition (Dissipative, m-dissipative and maximal dissipative operators)

A (possibly nonlinear) operator $N : D(N) \subset \mathbb{X} \to \mathbb{X}$ is called

(i) dissipative if $\langle z^*, N(x_1) - N(x_2) \rangle \leq 0$ for all $x_1, x_2 \in D(N)$ and some $z^* \in J(x_1 - x_2)$,

(ii) m-dissipative, if it is dissipative and $R(I - N) = \mathbb{X}$,

(iii) maximal dissipative, if it is dissipative and does not admit any proper dissipative extension.

The following proposition collects some properties of dissipative operators that will be used in the treatment of the HJBI equation.

Proposition

The following are true.

(i) *The operator $N : D(N) \subset \mathbb{X} \to \mathbb{X}$ is dissipative if and only if there exists a $\lambda > 0$, such that*

$$\|x_1 - x_2\|_{\mathbb{X}} \leq \frac{1}{\lambda}\|(\lambda x_1 - N(x_1)) - (\lambda x_2 - N(x_2))\|_{\mathbb{X}}, \quad \forall\, x_1, x_2 \in D(N). \tag{19}$$

If the above property holds for some $\lambda > 0$ then it holds for all $\lambda > 0$.

(ii) *The dissipative operator $N : D(N) \subset \mathbb{X} \to \mathbb{X}$ is m-dissipative if and only if there exists a $\lambda > 0$, such that $R(\lambda I - N) = \mathbb{X}$. If this property holds for some $\lambda > 0$, then it holds for all $\lambda > 0$.*

(iii) *Assume that the operator N_1 is m-dissipative on $\mathbb{X}$ and let N_2 be an operator with $\overline{\mathrm{dom}(N_1)} \subset \mathrm{dom}(N_2)$ which is continuous on $\overline{\mathrm{dom}(N_1)}$. If $N_1 + N_2$ is dissipative then it is m-dissipative.*

Based on the above, since we have a solution for $\delta > \delta_0$ so that $R(\delta I - \Gamma) = \mathbb{X}$, if Γ is dissipative then it is also m-dissipative so a solution exists for all $\delta > 0$.

It thus suffices to show the dissipativity of Γ.

This requires the use of a maximum priciple, which will allow us the comparison of two solutions of the HJBI equation for different choices of g_1, g_2.

Such a maximum principle can be obtained for finite dimensional approximations of the elliptic equation, which is an elliptic equation on $\mathbb{R}^n$ (Galerkin approximation)
, of the form
$\mathcal{A}\phi(x) = L_n\phi(x) := \langle A_n x, D\phi(x)\rangle_{\mathbb{R}^n} + \frac{1}{2}\mathrm{Tr}\left[C_n C_n^* D^2\phi(x)\right]$,
$x \in \mathbb{H}_n \simeq \mathbb{R}^n$, where $C_n C_n^*$ is the $n \times n$ covariance matrix obtained by the projection of the covariance operator CC^* to $\mathbb{H}_n$.

For such operators it can be proved (Cerrai 2000) that for any function $\phi \in \bigcap_{p\geq 1} W^{2,p}_{loc}(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)$ and any $\lambda > 0$ such that $(\lambda I - \mathcal{A})\phi \in C_b(\mathbb{R}^n)$, we have that

$$\|\phi\|_0 \leq \frac{1}{\lambda}\|\lambda\phi - \mathcal{A}\phi\|_0.$$

Use the finite dimensional approximation of the HJBI and consider two functions g_1, g_2.

For $\delta > \delta_0$ it holds that

$$\begin{aligned}\lambda(\phi_{1,n} - \phi_{2,n}) - L_n(\phi_{1,n} - \phi_{2,n}) = -(g_{1,n} - g_{2,n}) + \\ \langle F, D(\phi_{1,n} - \phi_{2,n}) + \mathbb{F}_\theta(D\phi_{1,n}) - \mathbb{F}_\theta(D\phi_{2,n}).\end{aligned} \qquad (20)$$

Since $\theta > \theta_0$, for any x there exists $q(x)$ with the property $\mathbb{F}(D\phi_{1,n}(x)) - \mathbb{F}(\phi_{2,n}(x)) \leq \langle q(x), D(\phi_{1,n}(x) - \phi_{2,n}(x))\rangle$, with $q(x)$ bounded. Substituting this estimate in (20) we obtain the inequality

$$\lambda(\phi_{1,n} - \phi_{2,n}) - L_{q,n}(\phi_{1,n} - \phi_{2,n}) \leq -(g_{1,n} - g_{2,n}), \qquad (21)$$

which implies

$$\|\lambda(\phi_{1,n} - \phi_{2,n}) - L_{q,n}(\phi_{1,n} - \phi_{2,n})\|_0 \leq \|g_{1,n} - g_{2,n}\|_0$$

On the other hand by the maximum principle for any $\phi_{i,n} \in \bigcap_{p\geq 1} W^{2,p}_{loc}(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)$ and any $\lambda > 0$ such that $(\lambda I - L_{q,n})\phi_{i,n} \in C_b(\mathbb{R}^n)$, $i = 1, 2$ it holds that

$$\|\phi_{1,n} - \phi_{2,n}\|_0 \leq \frac{1}{\lambda}\|\lambda(\phi_{1,n} - \phi_{2,n}) - L_{q,n}(\phi_{1,n} - \phi_{2,n})\|_0. \qquad (22)$$

We now pass to the limit as $n \to \infty$ in $\|\phi_{1,n} - \phi_{2,n}\|_0 \leq \frac{1}{\lambda}\|g_{1,n} - g_{2,n}\|_0$ taking into account that $\phi_{i,n} \to \phi_i$, where ϕ_i is the solution of $(\delta I - N)\phi_i = -g_i$, $i = 1, 2$.

Since by construction $g_{i,n} \to g_i$, $i = 1, 2$, we have that

$$\|\phi_1 - \phi_2\|_0 \leq \frac{1}{\lambda}\|g_1 - g_2\|_0 = \frac{1}{\lambda}\|\phi_1 - \phi_2 - \lambda(N(\phi_1) - N(\phi_2))\|_0, \quad (23)$$

where for the last equality we used the fact that ϕ_i solve $(\delta I - N)\phi_i = -g_i$, $i = 1, 2$.

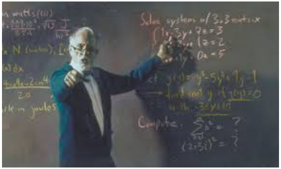
Inequality (23) is condition (19) for the operator N, hence the operator N is dissipative and this concludes the proof. QED

Therefore, if θ is such that condition (A) holds, we have a solution for all $\delta > 0$ by the m-dissipativity argument.

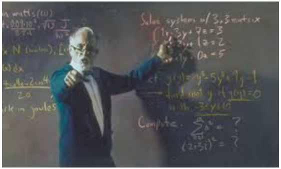
If θ is such that condition (A) does not hold, then, we only have solution for $\delta > \delta_0$ using the contraction mapping principle.

Condition (A) cannot hold if T satisfies a scaling law of the form $T^{\star}_{\theta}(p) = \theta^{-\alpha} T^{\star}(p)$, as $\theta \to 0$, for some $\alpha \geq 0$, where $T^{\star}$ is a proper lower semicontinuous convex function, then in the case where $\alpha > 0$, unless $\theta > \theta_0 > 0$!

Importantly for an entropic constraint this is the case with $\alpha > 0$ and the above result indicates possible breakdown of solutions of the robust control procedure in the limit of large uncertainty.



The next more useful thing to a solution



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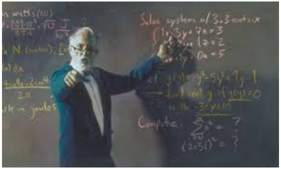
The next more useful thing to a solution



The next more useful thing to a solution

can be

its breakdown!

The value function and construction of optimal controls

Theorem

Let $(u^*(\cdot), \upsilon^*(\cdot)) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ be such that with $X^*(t) := X(x; u^*, v^*)$, the mild solution of (8) for the choice $u = u^*$ and $\upsilon = \upsilon^*$, and let V be the mild solution of the HJBI equation. Then, we have that

$$u^*(t) \in \mathrm{argmin}_{u \in \mathbb{H}} \Big[\langle u(t), -B^* DV(X^*(t)) \rangle_{\mathbb{H}} - K(u(t)) \Big],$$

$$\upsilon^*(t) \in \mathrm{argmax}_{\upsilon \in \mathbb{H}} \Big[\langle \upsilon(t), -C^* DV(X^*(t)) \rangle_{\mathbb{H}} - \theta T(\upsilon(t)) \Big],$$

a.e. $t \in [0, \infty)$, $\widetilde{\mathbb{P}} -$ a.s, with the above sets being singletons if K and T are strictly concave and convex respectively. Then, the triple $(X^*(\cdot), u^*(\cdot), \upsilon^*(\cdot))$ is optimal in the sense of (12). Furthermore, the optimal value function $\mathcal{V}$ of the stochastic differential game (10) coincides with the unique mild solution V of the HJBI equation (15)

This allows one to construct feedback control strategies which will provide the optimal control procedure in terms of the state of the system, leading to a closed feedback loop in the form

$$dX(t) = (AX(t) + F(x) + Bu^* + Cv^*)dt + Cd\widetilde{W}(t), \quad X(0) = x,$$

with u^*, v^* chosen such that

$$\begin{aligned} -\, u^*(X(t)) &\in \partial^+ K_\star(B^* DV(X(t)) \\ \upsilon^*(X(t)) &\in \partial T^\star_\theta \left(-\theta^{-1} C^* DV(X(t))\right), \end{aligned}$$

where V is the mild solution of the HJBI equation

Example : The spatial resource problem revisited

The biomass equation

$$\frac{\partial}{\partial t} y(t,z) = D\Delta y(t,z) + f(y(t,z)) - u(t,z) + \dot{W}(t,z),$$

will be understood as the stochastic SDE

$$dX(t) = (AX(t) + F(X(t)) - u(t))dt + CdW(t), \;\; X_0 = x,$$

on $\mathbb{H} = L^2(\mathcal{O})$, with

$$A = \Delta, \;\; D(A) = W_0^{1,2}(\mathcal{O}) \cap W^{2,2}(\mathcal{O})$$

and C is an operator such that CC^* corresponds to the spatial covariance structure of the stochastic fluctuations, driven by the Wiener process $\{W(t) \;:\; t \geq 0\}$ on $\mathbb{H}$.

Model uncertainty corresponds to different possible models for W, parameterized by different possible drifts $\upsilon \in \mathbb{H}$ and the payoff of the related differential game will be

$$J(x;u;\upsilon) = \int_0^\infty e^{-\delta t} \int_{\mathcal{O}} \left\{ g(y(t,z)) + \frac{1}{1-\nu}(u(t,z))^{1-\nu} + \frac{\theta}{p}|\upsilon(t,z)|^p \right\} dzdt,$$

Assuming that allowed controls are such that $0 \le u(x) \le R$, $R_1 \le \upsilon \le R_2$ we see that the value function for the game V is the mild solution of the HJBI equation

$$\delta V(x) - LV(x) = \langle F(x), DV(x)\rangle_{\mathbb{H}} + \Phi_1(DV(x)) + \Phi_2((C^* DV)(x)) + g(x),$$

for any $x \in \mathbb{H}$, with

$$\Phi_1(p) = \int_{\{z:\ p(z) < R^{-\nu}\}} \left(\frac{1}{1-\nu} R^{1-\nu} - Rp(z) \right) dz + \frac{\nu}{1-\nu} \int_{\{z:\ p(z) \ge R^{-\nu}\}} (p(z))^{-\frac{1-\nu}{\nu}} dz.$$

and

$$\Phi_2(q) = \int_{\{z:\ q(z) < -\theta R_2\}} \left(\frac{\theta}{2} R_2^2 + R_2 q(z) \right) dz - \int_{\{z:\ -\theta R_2 < q(z) < -\theta R_1\}} \frac{1}{2\theta} |q(z)|^2 dz + \int_{\{z:\ q(z) > -\theta R_1\}} \left(\frac{\theta}{2} R_1^2 + R_1 q(z) \right) dz.$$

The general results presented here guarantee the existence of a mild solution for the HJBI and having obtained this the optimal feedback laws are given in terms of

$$
\begin{aligned}
u^*(x(z)) &= R\,\mathbf{1}_{\{z\,:\,DV(x(z)<R^{-\nu}\}} \\
&\quad +(DV(x(z)))^{-1/\nu}\mathbf{1}_{\{z\,:\,DV(x(z)\geq R^{-\nu}\}}, \\
v^*(x(z)) &= R_1\,\mathbf{1}_{\{z\,:\,C^*DV(z)<\theta^2 R_1\}} \\
&+ \frac{1}{\theta^2}C^*DV(x(z))\mathbf{1}_{\{z\,:\,\theta^2 R_1\leq C^*DV(z)\leq\theta^2 R_2\}} \\
&\quad +R_2\mathbf{1}_{\{z\,:\,C^*DV(z)>\theta^2 R_2\}}
\end{aligned}
$$

The optimal state will be given by the solution of the SPDE closed loop system

$$d_tX(t,z) = (\Delta_z X(t,z) + F(X(t,z)) - u^*(X(t,z)) + Cv^*(X(t,z)))dt + CdW(t,z)$$

The limit as $\theta \to 0$ (deep uncertainty limit) is singular.

Conclusions and open problems

We have provided a framework for treating spatially extended stochastic optimal control systems under model uncertainty using dynamic programming.

The well posedeness of such problems and the characterization of the Nash equilibrium as well as the optimal control strategy is characterized in terms of mild solutions of nonlinear elliptic equations on Hilbert spaces.

This approach provides also interesting qualitative information on the nature of solutions, such as the breakdown of the robust control procedure in the limit of deep uncertainty ($\theta \to 0$).

Future challenges involve the numerical treatment of this problem based on the finite dimensional approximation of the HJBI equation used in the existence theorem and the generalization to more general fully nonlinear equations of the HJBI equation.

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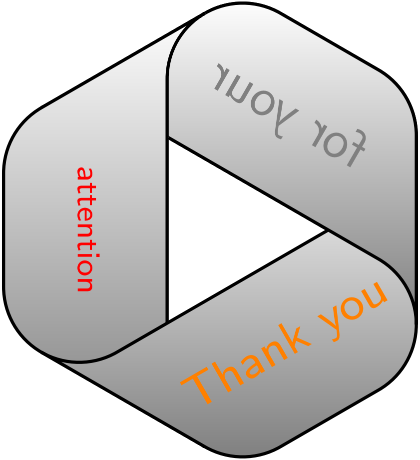
Thank you
for your
attention